Cones of weighted quasi-metrics, weighted quasi-hypermetrics and of oriented cuts

M.Deza V.Grishukhin E.Deza

Abstract

We show that the cone of weighted n-point quasi-metrics $WQMet_n$, the cone of weighted quasi-hypermetrics $WHyp_n$ and the cone of oriented cuts $OCut_n$ are projections along an extreme ray of the metric cone Met_{n+1}, of the hypermetric cone Hyp_{n+1} and of the cut cone Cut_{n+1}, respectively. This projection is such that if one knows all faces of an original cone then one knows all faces of the projected cone.

1 Introduction

Oriented (or directed) distances are encountered very often, for example, these are one-way transport routes, rivers with quick flow and so on.

The notions of directed distances, quasi-metrics and oriented cuts are generalizations of the notions of distances, metrics and cuts, respectively (see, for example, [DL97]), which are central objects in Graph Theory and Combinatorial Optimization.

Quasi-metrics are used in Semantics of Computations (see, for example, [Se97]) and in computational geometry (see, for example, [AACMP97]). Oriented distances have been used already by Hausdorff in 1914, see [Ha14].

In [CMM06], authors give an example of directed metric derived from a metric as follows. Let d be a metric on a set $V \cup \{0\}$, where 0 is a distinguished point. Then a quasi-metric q on the set V is given as

$$q_{ij} = d_{ij} + d_{i0} - d_{j0}.$$

This quasi-metric belongs to a special important subclass of quasi-metrics, namely, to a class of *weighted quasi-metrics*. We show (cf. also Lemma 1 (ii) in [DDV11]) that *any* weighted quasi-metric is obtained by a slight generalization of this method.

All semi-metrics on a set of cardinality n form a *metric cone* Met_n. There are two important sub-cones of Met_n, namely, the cone Hyp_n of *hypermetrics*, and the cone Cut_n of *ℓ_1-metrics*. These three cones form the following nested family $Cut_n \subset Hyp_n \subset Met_n$, see [DL97].

We introduce a space Q_n, called a *space of weighted quasi-metrics* and define in it a cone $WQMet_n$. Elements of this cone satisfy triangle and non-negativity inequalities. Among extreme rays of the cone $WQMet_n$ there are rays spanned by *ocut vectors*, i.e., incidence vectors of oriented cuts.

We define in the space Q_n a cone $OCut_n$ as the cone hull of ocut vectors. Elements of the cone $OCut_n$ are weighted quasi-ℓ-metrics.

Let semi-metrics in the cone Met_{n+1} be defined on the set $V \cup \{0\}$. The *cut cone* Cut_{n+1} (or the *cone of ℓ_1-metrics* on this set is a cone hull of cut semi-metrics $\delta(S)$ for all $S \subset V \cup \{0\}$. The cut semi-metrics $\delta(S)$ are extreme rays of all the three cones Met_{n+1}, Hyp_{n+1} and Cut_{n+1}. In particular, $\delta(\{0\}) = \delta(V)$ is an extreme ray of these three cones.

In this paper, it is shown that the cones $WQMet_n$ and $OCut_n$ are projections of the corresponding cones Met_{n+1} and Cut_{n+1} along the extreme ray $\delta(V)$. We define a cone $WQHyp_n$ of *weighted quasi-hypermetrics* as projection along $\delta(V)$ of the cone Hyp_{n+1}. So, we obtain a nested family $OCut_n \subset WQHyp_n \subset WQMet_n$.

The cones of weighted quasi-metrics, oriented cuts and other related generalizations of metrics are studied in [DD10] and [DDV11]. The polytope of oriented cuts was considered in [AM11].

2 Spaces $\mathbb{R}^E$ and $\mathbb{R}^{E^{\mathcal{O}}}$

Let V be a set of cardinality $|V| = n$. Let E and $E^{\mathcal{O}}$ be sets of all unordered (ij) and ordered ij pairs of elements $i, j \in V$. Consider two Euclidean spaces $\mathbb{R}^E$ and $\mathbb{R}^{E^{\mathcal{O}}}$ of vectors $d \in \mathbb{R}^E$ and $g \in \mathbb{R}^{E^{\mathcal{O}}}$ with coordinates $d_{(ij)}$ and g_{ij}, where $(ij) \in E$ and $ij \in E^{\mathcal{O}}$, respectively. Obviously, dimensions of the spaces $\mathbb{R}^E$ and $\mathbb{R}_s^{E^{\mathcal{O}}}$ are $|E| = \frac{n(n-1)}{2}$ and $|E^{\mathcal{O}}| = n(n-1)$, respectively.

Denote by $(d, t) = \sum_{(ij)\in E} d_{(ij)} t_{(ij)}$ scalar product of vectors $d, t \in \mathbb{R}^E$. Similarly, $(f, g) = \sum_{ij\in E^{\mathcal{O}}} f_{ij} g_{ij}$ is the scalar product of vectors $f, g \in \mathbb{R}^{E^{\mathcal{O}}}$.

Let $\{e_{(ij)} : (ij) \in E\}$ and $\{e_{ij} : ij \in E^{\mathcal{O}}\}$ be orthonormal bases of $\mathbb{R}^E$ and $\mathbb{R}^{E^{\mathcal{O}}}$, respectively. Then, for $f \in \mathbb{R}^E$ and $q \in \mathbb{R}^{E^{\mathcal{O}}}$, we have

$$(e_{(ij)}, f) = f_{(ij)} \text{ and } (e_{ij}, q) = q_{ij}.$$

For $f \in \mathbb{R}^{E^{\mathcal{O}}}$, define $f^* \in \mathbb{R}^{E^{\mathcal{O}}}$ as follows

$$f^*_{ij} = f_{ji} \text{ for all } ij \in E^{\mathcal{O}}.$$

Each vector $g \in \mathbb{R}^{E^{\mathcal{O}}}$ can be decompose into *symmetric* g^s and *antisymmetric* g^a parts as follows:

$$g^s = \frac{1}{2}(g + g^*),\ g^a = \frac{1}{2}(g - g^*),\ g = g^s + g^a.$$

Call a vector g *symmetric* if $g^* = g$, and *antisymmetric* if $g^* = -g$. Let $\mathbb{R}_s^{E^{\mathcal{O}}}$ and $\mathbb{R}_a^{E^{\mathcal{O}}}$ be subspaces of the corresponding vectors. Note that the spaces $\mathbb{R}_s^{E^{\mathcal{O}}}$ and $\mathbb{R}_a^{E^{\mathcal{O}}}$ are mutually orthogonal. In fact, for $p \in \mathbb{R}_s^{E^{\mathcal{O}}}$ and $f \in \mathbb{R}_a^{E^{\mathcal{O}}}$, we have

$$(p, f) = \sum_{ij\in E^{\mathcal{O}}} p_{ij} f_{ij} = \sum_{(ij)\in E} (p_{ij} f_{ij} + p_{ji} f_{ji}) = \sum_{(ij)\in E} (p_{ij} f_{ij} - p_{ij} f_{ij}) = 0.$$

Hence

$$\mathbb{R}^{E^{\mathcal{O}}} = \mathbb{R}_s^{E^{\mathcal{O}}} \oplus \mathbb{R}_a^{E^{\mathcal{O}}},$$

where $\oplus$ is the direct sum.

Obviously, there is an isomorphism φ between the spaces $\mathbb{R}^E$ and $\mathbb{R}_s^{E^{\mathcal{O}}}$. Let $d \in \mathbb{R}^E$ have coordinates $d_{(ij)}$. Then

$$d^{\mathcal{O}} = \varphi(d) \in \mathbb{R}_s^{E^{\mathcal{O}}} \text{ , such that } d_{ij}^{\mathcal{O}} = d_{ji}^{\mathcal{O}} = d_{(ij)}.$$

In particular,

$$\varphi(e_{(ij)}) = e_{ij} + e_{ji}.$$

The map φ is invertible. In fact, for $q \in \mathbb{R}_s^{E^{\mathcal{O}}}$, we have $\varphi^{-1}(q) = d \in \mathbb{R}^E$, such that $d_{(ij)} = q_{ij} = q_{ji}$. The isomorphism φ will be useful in what follows.

3 Space of weights Q_n^w

One can consider the sets E and $E^{\mathcal{O}}$ as sets of edges (ij) and arcs ij of an unordered and ordered complete graphs K_n and $K_n^{\mathcal{O}}$ on the vertex set V, respectively. The graph $K_n^{\mathcal{O}}$ has two arcs ij and ji between each pair of vertices $i, j \in V$.

It is convenient to consider vectors $g \in \mathbb{R}^{E^{\mathcal{O}}}$ as functions on the set of arcs $E^{\mathcal{O}}$ of the graph $K_n^{\mathcal{O}}$. So, the decomposition $\mathbb{R}^{E^{\mathcal{O}}} = \mathbb{R}_s^{E^{\mathcal{O}}} \oplus \mathbb{R}_a^{E^{\mathcal{O}}}$ is a decomposition of the space of all functions on arcs in $E^{\mathcal{O}}$ onto the spaces of symmetric and antisymmetric functions.

Besides, there is an important direct decomposition of the space $\mathbb{R}_a^{E^{\mathcal{O}}}$ of antisymmetric functions into two subspaces. In the Theory of Electric Networks, these spaces are called spaces of *tensions* and *flows* (see also [Aig79]).

The tension space relates to *potentials* (or *weights*) w_i given on vertices $i \in V$ of the graph $K_n^{\mathcal{O}}$. The corresponding antisymmetric function g^w is determined as

$$g_{ij}^w = w_i - w_j.$$

It is called *tension* on the arc ij. Obviously, $g_{ji}^w = w_j - w_i = -g_{ij}^w$. Denote by Q_n^w the subspace of $\mathbb{R}^{E^{\mathcal{O}}}$ generated by all tensions on arcs $ij \in E^{\mathcal{O}}$. We call Q_n^w a *space of weights*.

Each tension function g^w is represented as weighted sum of elementary *potential* functions $q(k)$, for $k \in V$, as follows:

$$g^w = \sum_{k \in V} w_k q(k),$$

where

$$q(k) = \sum_{j \in V - \{k\}} (e_{kj} - e_{jk}), \text{ for all } k \in V, \tag{1}$$

are basic functions that generate the space of weights Q_n^w. Hence, the values of the basic functions $q(k)$ on arcs are as follows:

$$q_{ij}(k) = \begin{cases} 1, & \text{if } i = k \\ -1, & \text{if } j = k \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

We obtain

$$g_{ij}^w = \sum_{k\in V} w_k q_{ij}(k) = w_i - w_j.$$

It is easy to verify that

$$q^2(k) = (q(k), q(k)) = 2(n-1),\ (q(k), q(l)) = -2 \text{ for all } k, l \in V, k \neq l,\ \sum_{k\in V} q(k) = 0.$$

Hence, there are only $n-1$ independent functions $q(k)$ that generate the space Q_n^w.

The weighted quasi-metrics lie in the space $\mathbb{R}_s^{E^{\mathcal{O}}} \oplus Q_n^w$ that we denote as Q_n. Direct complements of Q_n^w in $\mathbb{R}_a^{E^{\mathcal{O}}}$ and Q_n in $\mathbb{R}^{E^{\mathcal{O}}}$ is a space Q_n^c of *circuits* (or *flows*).

4 Space of circuits Q_n^c

The *space of circuits* (or *space of flows*) is generated by characteristic vectors of oriented circuits in the graph $K_n^{\mathcal{O}}$. Arcs of $K_n^{\mathcal{O}}$ are ordered pairs ij of vertices $i, j \in V$. The arc ij is oriented from the vertex i to the vertex j. Recall that $K_n^{\mathcal{O}}$ has both the arcs ij and ji for each pair of vertices $i, j \in V$.

Let $G_s \subset K_n$ be a subgraph with a set of edges $E(G_s) \subset E$. We relate to the graph G_s a directed graph $G \subset K_n^{\mathcal{O}}$ with the arc set $E^{\mathcal{O}}(G) \subset E^{\mathcal{O}}$ as follows. An arc ij belongs to G, i.e., $ij \in E^{\mathcal{O}}(G)$, if and only if $(ij) = (ji) \in E(G)$. This definition implies that in this case, the arc ji belongs to G also, i.e., $ji \in E^{\mathcal{O}}(G)$.

Let C_s be a circuit in the graph K_n. The circuit C_s is determined by a sequence of distinct vertices $i_k \in V$, where $1 \leq k \leq p$, and p is the length of C_s. The edges of C_s are unordered pairs (i_k, i_{k+1}), where indices are taken modulo p. By above definition, an *oriented bicircuit* C of the graph $K_n^{\mathcal{O}}$ relates to the circuit C_s. Arcs of C are ordered pairs $i_k i_{k+1}$ and $i_{k+1} i_k$, where indices are taken modulo p. Take an orientation of C. Denote by $-C$ the *opposite* circuit with opposite orientation. Denote an arc of C *direct* or *opposite* if its direction coincides with or is opposite to the given orientation of C, respectively. Let C^+ and C^- be subcircuits of C consisting of direct and opposite arcs, respectively.

The following vector f^C is the characteristic vector of the bicircuit C:

$$f_{ij}^C = \begin{cases} 1, & \text{if } ij \in C^+, \\ -1. & \text{if } ij \in C^-, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $f^{-C} = (f^C)^* = -f^C$, and $f^C \in \mathbb{R}_a^{E^{\mathcal{O}}}$.

Denote by Q_n^c the space linearly generated by circuit vectors f^C for all bicircuits C of the graph $K_n^{\mathcal{O}}$. It is well known that characteristic vectors of *fundamental* circuits form a basis of Q_n^c. Fundamental circuits are defined as follows.

Let T be a spanning tree of the graph K_n. Since T is spanning, its vertex set $V(T)$ is the set of all vertices of K_n, i.e., $V(T) = V$. Let $E(T) \subset E$ be the set of edges of T. Then any edge $e = (ij) \notin E(T)$ closes a unique path in T between vertices i and j into a circuit C_s^e. This circuit C_s^e is called *fundamental*. Call corresponding oriented bicircuit C^e also *fundamental*.

There are $|E - E(T)| = \frac{n(n-1)}{2} - (n-1)$ fundamental circuits. Hence

$$\dim Q_n^c = \frac{n(n-1)}{2} - (n-1), \text{ and } \dim Q_n + \dim Q_n^c = n(n-1) = \dim \mathbb{R}^{E^{\mathcal{O}}}.$$

This implies that Q_n^c is an orthogonal complement of Q_n^w in $\mathbb{R}_a^{\mathcal{O}}$ and Q_n in $\mathbb{R}^{E^{\mathcal{O}}}$, i.e.

$$\mathbb{R}_a^{E^{\mathcal{O}}} = Q_n^w \oplus Q_n^c \text{ and } \mathbb{R}^{E^{\mathcal{O}}} = Q_n \oplus Q_n^c = \mathbb{R}^{E_s^{\mathcal{O}}} \oplus Q_n^w \oplus Q_n^c.$$

5 Cut and ocut vector set-functions

The space Q_n is generated also by vectors of oriented cuts, which we define in this section.

Each subset $S \subset V$ determines cuts of the graphs K_n and $K_n^{\mathcal{O}}$ that are subsets of edges and arcs of these graphs.

A $cut(S) \subset E$ is a subset of edges (ij) of K_n such that $(ij) \in cut(S)$ if and only if $|\{i,j\} \cap S| = 1$.

A $cut^{\mathcal{O}}(S) \subset E^{\mathcal{O}}$ is a subset of arcs ij of $K_n^{\mathcal{O}}$ such that $ij \in cut^{\mathcal{O}}(S)$ if and only if $|\{i,j\} \cap S| = 1$. So, if $ij \in cut^{\mathcal{O}}(S)$, then $ji \in cut^{\mathcal{O}}(S)$ also.

An *oriented cut* is a subset $ocut(S) \subset E^{\mathcal{O}}$ of arcs ij of $K_n^{\mathcal{O}}$ such that $ij \in ocut(S)$ if and only if $i \in S$ and $j \notin S$.

We relate to these three types of cuts characteristic vectors $\delta(S) \in \mathbb{R}^E$, $\delta^{\mathcal{O}}(S) \in \mathbb{R}_s^{E^{\mathcal{O}}}$, $q(S) \in \mathbb{R}_a^{E^{\mathcal{O}}}$ and $c(S) \in \mathbb{R}^{E^{\mathcal{O}}}$ as follows.

For $cut(S)$, we set

$$\delta(S) = \sum_{i \in S, j \in \overline{S}} e_{(ij)}, \text{ such that } \delta_{(ij)}(S) = \begin{cases} 1, & \text{if } |\{i,j\} \cap S| = 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $\overline{S} = V - S$. For $cut^{\mathcal{O}}(S)$, we set

$$\delta^{\mathcal{O}}(S) = \varphi(\delta(S)) = \sum_{i \in S, j \in \overline{S}} (e_{ij} + e_{ji}) \text{ and } q(S) = \sum_{i \in S, j \in \overline{S}} (e_{ij} - e_{ji}).$$

Hence,

$$\delta_{ij}^{\mathcal{O}}(S) = \begin{cases} 1, & \text{if } |\{i,j\} \cap S| = 1 \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } q_{ij}(S) = \begin{cases} 1, & \text{if } i \in S, j \notin S \\ -1, & \text{if } j \in S, i \notin S \\ 0, & \text{otherwise.} \end{cases}$$

Note that, for one-element sets $S = \{k\}$, the function $q(\{k\})$ is $q(k)$ of section 2. It is easy to see that

$$(\delta^{\mathcal{O}}(S), q(T)) = 0 \text{ for any } S, T \subset V.$$

For the oriented cut $ocut(S)$, we set

$$c(S) = \sum_{i \in S, j \in \overline{S}} e_{ij}.$$

Hence,

$$c_{ij}(S) = \begin{cases} 1, & \text{if } i \in S, j \notin S \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, it holds $c(\emptyset) = c(V) = \mathbf{0}$, where $\mathbf{0} \in \mathbb{R}^{E^{\mathcal{O}}}$ is a vector whose all coordinates are equal zero. We have

$$c^*(S) = c(\overline{S}),\ c(S) + c(\overline{S}) = \delta^{\mathcal{O}}(S),\ c(S) - c(\overline{S}) = q(S) \text{ and } c(S) = \frac{1}{2}(\delta^{\mathcal{O}}(S) + q(S)). \quad (3)$$

Besides, we have

$$c^s(S) = \frac{1}{2}\delta^{\mathcal{O}}(S),\ c^a(S) = \frac{1}{2}q(S).$$

Recall that a set-function $f(S)$ on all $S \subset V$, is called *submodular* if, for any $S, T \subset V$, the following *submodular inequality* holds:

$$f(S) + f(T) - (f(S \cap T) + f(S \cup T)) \geq 0.$$

It is well known that the vector set-function $\delta \in \mathbb{R}^E$ is submodular (see, for example, [Aig79]). The above isomorphism φ of the spaces $\mathbb{R}^E$ and $\mathbb{R}^{E^{\mathcal{O}}}_s$ implies that the vector set-function $\delta^{\mathcal{O}} = \varphi(\delta) \in \mathbb{R}^{E^{\mathcal{O}}}_s$ is submodular also.

A set-function $f(S)$ is called *modular* if, for any $S, T \subset V$, the above submodular inequality holds as equality. This equality is called *modular equality.* It is well known (and can be easily verified) that antisymmetric vector set-function $f^a(S)$ is modular for any oriented graph G. Hence, our antisymmetric vector set-function $q(S) \in \mathbb{R}^{E^{\mathcal{O}}}_a$ for the oriented complete graph $K^{\mathcal{O}}_n$ is modular also.

Note that the set of all submodular set-functions on a set V forms a cone in the space $\mathbb{R}^{2^V}$. Therefore, the last equality in (3) implies that the vector set-function $c(S) \in \mathbb{R}^{E^{\mathcal{O}}}$ is submodular.

The modularity of the antisymmetric vector set-function $q(S)$ is important for what follows. It is well-known (see, for example, [Bir67]) (and it can be easily verified using modular equality) that a modular set-function $m(S)$ is completely determined by its values on the empty set and on all one-element sets. Hence, a modular set-function $m(S)$ has the following form

$$m(S) = m_0 + \sum_{i \in S} m_i,$$

where $m_0 = m(\emptyset)$ and $m_i = m(\{i\}) - m(\emptyset)$. For brevity, we set $f(\{i\}) = f(i)$ for any set-function $f(S)$. Since $q(\emptyset) = q(V) = 0$, we have

$$q(S) = \sum_{k \in S} q(k),\ S \subset V, \text{ and } q(V) = \sum_{k \in V} q(k) = 0. \quad (4)$$

Using equations (3) and (4), we obtain

$$c(S) = \frac{1}{2}(\delta^{\mathcal{O}}(S) + \sum_{k \in S} q(k)). \quad (5)$$

Now we show that ocut vectors $c(S)$ for all $S \subset V$ linearly generate the space $Q_n \subseteq \mathbb{R}^{E^{\mathcal{O}}}$. The space generated by $c(S)$ consists of the following vectors

$$c = \sum_{S \subset V} \alpha_S c(S), \text{ where } \alpha_S \in \mathbb{R}.$$

Recall that $c(S) = \frac{1}{2}(\delta^{\mathcal{O}}(S) + q(S))$. Hence, we have

$$c = \frac{1}{2} \sum_{S \subset V} \alpha_S(\delta^{\mathcal{O}}(S) + q(S)) = \frac{1}{2} \sum_{S \subset V} \alpha_S \delta^{\mathcal{O}}(S) + \frac{1}{2} \sum_{S \subset V} \alpha_S q(S) = \frac{1}{2}(d^{\mathcal{O}} + q),$$

where $d^{\mathcal{O}} = \varphi(d)$ for $d = \sum_{S \subset V} \alpha_S \delta(S)$. For a vector q, we have

$$q = \sum_{S \subset V} \alpha_S q(S) = \sum_{S \subset V} \alpha_S \sum_{k \in S} q(k) = \sum_{k \in V} w_k q(k), \text{ where } w_k = \sum_{V \supset S \ni k} \alpha_S.$$

Since $q_{ij} = \sum_{k \in V} w_k q_{ij}(k) = w_i - w_j$, we have

$$c_{ij} = \frac{1}{2}(d^{\mathcal{O}}_{ij} + w_i - w_j). \tag{6}$$

It is well-known (see, for example, [DL97]) that the cut vectors $\delta(S) \in \mathbb{R}^E$ for all $S \subset V$ linearly generate the full space $\mathbb{R}^E$. Hence, the vectors $\delta^{\mathcal{O}}(S) \in \mathbb{R}^{E^{\mathcal{O}}}_s$, for all $S \subset V$, linearly generate the full space $\mathbb{R}^{E^{\mathcal{O}}}_s$.

According to (2), antisymmetric parts of ocut vectors $c(S)$ generate the space Q^w_n. This implies that the space $Q_n = \mathbb{R}^{E^{\mathcal{O}}}_s \oplus Q^w_n$ is generated by $c(S)$ for all $S \subset V$.

6 Properties of the space Q_n

Let $x \in Q_n$ and let f^C be the characteristic vector of a bicircuit C. Since f^C is orthogonal to Q_n, we have $(x, f^C) = \sum_{ij \in C} f^C_{ij} x_{ij} = 0$. This equality implies that each point $x \in Q_n$ satisfies the following equalities

$$\sum_{ij \in C^+} x_{ij} = \sum_{ij \in C^-} x_{ij}$$

for any bicircuit C.

Let $K_{1,n-1} \subset K_n$ be a spanning star of K_n consisting of all $n-1$ edges incident to a vertex of K_n. Let this vertex be 1. Each edge of $K_n - K_{1,n-1}$ has the form (ij), where $i \neq 1 \neq j$. The edge (ij) closes a fundamental triangle with edges $(1i), (1j), (ij)$. The corresponding bitriangle $T(1ij)$ generates the equality

$$x_{1i} + x_{ij} + x_{j1} = x_{i1} + x_{1j} + x_{ji}.$$

These equalities are the case $k = 3$ of *k-cyclic symmetry* considered in [DD10]. They were derived by another way in [AM11]. They correspond to fundamental bi-triangles $T(1ij)$, for

all $i, j \in V - \{1\}$, and are all $\frac{n(n-1)}{2} - (n-1)$ independent equalities determining the space, where the Q_n lies.

Above coordinates x_{ij} of a vector $x \in Q_n$ are given in the orthonormal basis $\{e_{ij} : ij \in E^{\mathcal{O}}\}$. But, for what follows, it is more convenient to consider vectors $q \in Q_n$ in another basis. Recall that $\mathbb{R}_s^{E^{\mathcal{O}}} = \varphi(\mathbb{R}^E)$. Let, for $(ij) \in E$, $\varphi(e_{(ij)}) = e_{ij} + e_{ji} \in \mathbb{R}_s^{E^{\mathcal{O}}}$ be basic vectors of the subspace $\mathbb{R}_s^{E^{\mathcal{O}}} \subset Q_n$. Let $q(i) \in Q_n^w$, $i \in V$, be basic vectors of the space $Q_n^w \subset Q_n$. Then, for $q \in Q_n$, we set

$$q = q^s + q^a, \text{ where } q^s = \sum_{(ij) \in E} q_{(ij)} \varphi(e_{(ij)}), \ q^a = \sum_{i \in V} w_i q(i).$$

Now, we obtain an important expression for the scalar product (g, q) of vectors $g, q \in Q_n$. Recall that $(\varphi(e_{(ij)}), q(k)) = ((e_{ij} + e_{ji}), q(k)) = 0$ for all $(ij) \in E$ and all $k \in V$. Hence $(g^s, q^a) = (g^a, q^s) = 0$, and we have

$$(g, q) = (g^s, q^s) + (g^a, q^a).$$

Besides, we have

$$((e_{ij} + e_{ji}), (e_{kl} + e_{lk})) = 0 \text{ if } (ij) \neq (kl), \ (e_{ij} + e_{ji})^2 = 2,$$

and (see Section 3)

$$(q(i), q(j)) = -2 \text{ if } i \neq j, \ (q(i))^2 = 2(n-1).$$

Let v_i, $i \in V$, be weights of the vector g. Then we have

$$(g, q) = 2 \sum_{(ij) \in E} g_{(ij)} q_{(ij)} + 2(n-1) \sum_{i \in V} v_i w_i - 2 \sum_{i \neq j \in V} v_i w_j.$$

For the last sum, we have

$$\sum_{i \neq j \in V} v_i w_j = (\sum_{i \in V} v_i)(\sum_{i \in V} w_i) - \sum_{i \in V} v_i w_i.$$

Since weights are defined up to an additive scalar, we can choose weights v_i such that $\sum_{i \in V} v_i = 0$. Then the last sum in the product (g, q) is equal to $-\sum_{i \in V} v_i w_i$. Finally we obtain that the sum of antisymmetric parts is equal to $2n \sum_{i \in V} v_i w_i$. So, for the product of two vectors $g, q \in Q_n$ we have the following expression

$$(g, q) = (g^s, q^s) + (g^a, q^a) = 2(\sum_{(ij) \in E} g_{(ij)} q_{(ij)} + n \sum_{i \in V} v_i w_i) \text{ if } \sum_{i \in V} v_i = 0 \text{ or } \sum_{i \in V} w_i = 0.$$

In what follows, we consider inequalities $(g, q) \geq 0$. We can delete the multiple 2, and rewrite such inequality as follows

$$\sum_{(ij) \in E} g_{(ij)} q_{(ij)} + n \sum_{i \in V} v_i w_i \geq 0, \tag{7}$$

where $\sum_{i \in V} v_i = 0$.

Below we consider some cones in the space Q_n. Since the space Q_n is orthogonal to the space of circuits Q_n^c, each facet vector of a cone in Q_n is defined up to a vector of the space Q_n^c. Of course each vector $g' \in \mathbb{R}^{E^{\mathcal{O}}}$ can be decomposed as $g' = g + g^c$, where $g \in Q_n$ and $g^c \in Q_n^c$. Call the vector $g \in Q_n$ *canonical representative* of the vector g'. Usually we will use canonical facet vectors. But sometimes not canonical representatives of a facet vector are useful.

Cones Con that will be considered are invariant under the operation $q \to q^*$, defined in Section 2. In other words, $Con^* = Con$. This operation changes signs of weights:

$$q_{ij} = q_{(ij)} + w_i - w_j \to q_{(ij)} + w_j - w_i = q_{(ij)} - w_i + w_j.$$

Let $(g, q) \geq 0$ be an inequality determining a facet F of a cone $Con \subset Q_n$. Since $Con = Con^*$, the cone Con has, together with the facet F, also a facet F^*. The facet F^* is determined by the inequality $(g^*, q) \geq 0$.

7 Projections of cones Con_{n+1}

Recall that $Q_n = \mathbb{R}_s^{E^{\mathcal{O}}} \oplus Q_n^w$, $\mathbb{R}_s^{E^{\mathcal{O}}} = \varphi(\mathbb{R}^E)$ and $\dim Q_n = \frac{n(n+1)}{2} - 1$. Let $0 \notin V$ be an additional point. Then the set of unordered pairs (ij) for $i, j \in V \cup \{0\}$ is $E \cup E_0$, where $E_0 = \{(0i) : i \in V\}$. Obviously, $\mathbb{R}^{E \cup E_0} = \mathbb{R}^E \oplus \mathbb{R}^{E_0}$ and $\dim \mathbb{R}^{E \cup E_0} = \frac{n(n+1)}{2}$.

The space $\mathbb{R}^{E \cup E_0}$ contains the following three important cones: the cone Met_{n+1} of semi-metrics, the cone Hyp_{n+1} of hyper-semi-metrics and the cone Cut_{n+1} of ℓ_1-semi-metrics, all on the set $V \cup \{0\}$. Denote by Con_{n+1} any of these cones.

Recall that a semi-metric $d = \{d_{(ij)}\}$ is called *metric* if $d_{(ij)} \neq 0$ for all $(ij) \in E$. For brevity sake, in what follows, we call elements of the cones Con_{n+1} simply metrics (or hypermetrics, ℓ_1-metrics), assuming that they can be semi-metrics.

Note that if $d \in Con_{n+1}$ is a metric on the set $V \cup \{0\}$, then a restriction d^V of d on the set V is a point of the cone $Con_n = Con_{n+1} \cap \mathbb{R}^E$ of metrics on the set V. In other words, we can suppose that $Con_n \subset Con_{n+1}$.

The cones Met_{n+1}, Hyp_{n+1} and Cut_{n+1} contain the cut vectors $\delta(S)$ that span extreme rays for all $S \subset V \cup \{0\}$. Denote by l_0 the extreme ray spanned by the cut vector $\delta(V) = \delta(\{0\})$. Consider a projection $\pi(\mathbb{R}^{E \cup E_0})$ of the space $\mathbb{R}^{E \cup E_0}$ along the ray l_0 onto a subspace of $\mathbb{R}^{E \cup E_0}$ that is orthogonal to $\delta(V)$. This projection is such that $\pi(\mathbb{R}^E) = \mathbb{R}^E$ and $\pi(\mathbb{R}^{E \cup E_0}) = \mathbb{R}^E \oplus \pi(\mathbb{R}^{E_0})$.

Note that $\delta(V) \in \mathbb{R}^{E_0}$, since, by Section 5, $\delta(V) = \sum_{i \in V} e_{(0i)}$. For simplicity sake, set

$$e_0 = \delta(\{0\}) = \delta(V) = \sum_{i \in V} e_{(0i)}.$$

Recall that the vector e_0 spans the extreme ray l_0. Obviously, the space $\mathbb{R}^E$ is orthogonal to l_0, and therefore, $\pi(\mathbb{R}^E) = \mathbb{R}^E$.

Let $x \in \mathbb{R}^E$. We decompose this point as follows

$$x = x^V + x^0,$$

where $x^V = \sum_{(ij)\in E} x_{(ij)}e_{(ij)} \in \mathbb{R}^E$ and $x^0 = \sum_{i\in V} x_{(0i)}e_{(0i)} \in \mathbb{R}^{E_0}$. We define a map π as follows:

$$\pi(e_{(ij)}) = e_{(ij)} \text{ for } (ij) \in E, \text{ and } \pi(e_{(0i)}) = e_{(0i)} - \frac{1}{n}e_0 \text{ for } i \in V.$$

So, we have

$$\pi(x) = \pi(x^V) + \pi(x^0) = \sum_{(ij)\in E} x_{(ij)}e_{(ij)} + \sum_{i\in V} x_{(0i)}(e_{(0i)} - \frac{1}{n}e_0). \tag{8}$$

Note that the projection π transforms the positive orthant of the space $\mathbb{R}^{E_0}$ onto the whole space $\pi(\mathbb{R}^{E_0})$.

Now we describe how faces of a cone in the space $\mathbb{R}^{E\cup E_0}$ are projected along one of its extreme rays.

Let l be an extreme ray and F be a face of a cone in $\mathbb{R}^{E\cup E_0}$. Let π be the projection along l. Let $\dim F$ be dimension of the face F. Then the following equality holds

$$\dim\pi(F) = \dim F - \dim(F \cap l). \tag{9}$$

Let $g \in \mathbb{R}^{E\cup E_0}$ be a facet vector of a facet G, and e be a vector spanning the line l. Then $\dim(G \cap l) = 1$ if $(g, e) = 0$, and $\dim(G \cap l) = 0$ if $(g, e) \neq 0$.

Theorem 1 *Let G be a face of the cone $\pi(Con_{n+1})$. Then $G = \pi(F)$, where F is a face of Con_{n+1} such that there is a facet of Con_{n+1}, containing F and the extreme ray l_0 spanned by $e_0 = \delta(V)$.*

In particular, G is a facet of $\pi(Con_{n+1})$ if and only if $G = \pi(F)$, where F is a facet of Con_{n+1} containing the extreme ray l_0. Similarly, l' is an extreme ray of $\pi(Con_{n+1})$ if and only if $l' = \pi(l)$, where l is an extreme ray of Con_{n+1} lying in a facet of Con_{n+1} that contains l_0.

Proof. Let $\mathcal{F}$ be a set of all facets of the cone Con_{n+1}. Then $\cup_{f\in\mathcal{F}}\pi(F)$ is a covering of the projection $\pi(Con_{n+1})$. By (9), in this covering, if $l_0 \subset F \in \mathcal{F}$, then $\pi(F)$ is a facet of $\pi(Con_{n+1})$. If $l_0 \not\subset F$, then there is a one-to-one correspondence between points of F and $\pi(F)$. Hence, $\dim\pi(F) = n$, and $\pi(F)$ cannot be a facet of $\pi(Con_{n+1})$, since $\pi(F)$ fills an n-dimensional part of the cone $\pi(Con_{n+1})$.

If F' is a face of Con_{n+1}, then $\pi(F')$ is a face of the above covering. If F' belongs only to facets $F \in \mathcal{F}$ such that $l_0 \not\subset F$, then $\pi(F')$ lies inside of $\pi(Con_{n+1})$. In this case, it is not a face of $\pi(Con_{n+1})$. This implies that $\pi(F')$ is a face of $\pi(Con_{n+1})$ if and only if $F' \subset F$, where F is a facet of Con_{n+1} such that $l_0 \subset F$. Suppose that dimension of F' is $n-1$, and $l_0 \not\subset F'$. Then $\dim\pi(F') = n-1$. If F' is contained in a facet F of Con_{n+1} such that $l_0 \subset F$, then $\pi(F') = \pi(F)$. Hence, $\pi(F')$ is a facet of the cone $\pi(Con_{n+1})$ that coincides with the facet $\pi(F)$.

Now, the assertions of Theorem about facets and extreme rays of $\pi(Con_{n+1})$ follow. □

Theorem 1 describes all faces of the cone $\pi(Con_{n+1})$ if one knows all faces of the cone Con_{n+1}.

Recall that we consider $Con_n = Con_{n+1} \cap \mathbb{R}^E$ as a sub-cone of Con_{n+1}, and therefore, $\pi(Con_n) \subset \pi(Con_{n+1})$. Since $\pi(\mathbb{R}^E) = \mathbb{R}^E$, we have $\pi(Con_n) = Con_n$. Let $(f, x) \geq$ be a facet-defining inequality of a facet F of the cone Con_{n+1}. Since $Con_{n+1} \subset \mathbb{R}^E \oplus \mathbb{R}^{E_0}$, we represent vectors $f, x \in \mathbb{R}^{E \cup E_0}$ as $f = f^V + f^0, x = x^V + x^0$, where $f^V, x^V \in \mathbb{R}^E$ and $f^0, x^0 \in \mathbb{R}^{E_0}$. Hence, the above facet-defining inequality can be rewritten as

$$(f, x) = (f^V, x^V) + (f^0, x^0) \geq 0.$$

It turns out that Con_{n+1} has always a facet F with its facet vector $f = f^V + f^0$ such that $f^0 = 0$. Since f^V is orthogonal to $\mathbb{R}^{E_0}$, the hyperplane $(f^V, x) = (f^V, x^V) = 0$ supporting the facet F contains the whole space $\mathbb{R}^{E_0}$. The equality $(f^V, x^V) = 0$ defines a facet $F^V = F \cap \mathbb{R}^E$ of the cone Con_n.

Definition. A facet F of the cone Con_{n+1} with a facet vector $f = f^V + f^0$ is called *zero-lifting* of a facet F^V of Con_n if $f^0 = 0$ and $F \cap \mathbb{R}^E = F^V$.

Similarly, a facet $\pi(F)$ of the cone $\pi(Con_{n+1})$ with a facet vector f is called zero-lifting of F^V if $f = f^V$ and $\pi(F) \cap \mathbb{R}^E = F^V$.

It is well-known (see, for example, [DL97]) that each facet F^V with facet vector f^V of the cone Con_n can be zero-lifted up to a facet F of Con_{n+1} with the same facet vector f^V.

Proposition 1 *Let a facet F of Con_{n+1} be a zero-lifting of a facet F^V of Con_n. Then $\pi(F)$ is a facet of $\pi(Con_{n+1})$ that is also zero-lifting of F^V.*

Proof. Recall that the hyperplane $\{x \in \mathbb{R}^{E \cup E_0} : (f^V, x) = 0\}$ supporting the facet F contains the whole space $\mathbb{R}^{E_0}$. Hence, the facet F contains the extreme ray l_0 spanned by the vector $e_0 \in \mathbb{R}^{E_0}$. By Theorem 1, $\pi(F)$ is a facet of $\pi(Con_{n+1})$. The facet vector of $\pi(F)$ can be written as $f = f^V + f'$, where $f^V \in \mathbb{R}^E$ and $f' \in \pi(\mathbb{R}^{E_0})$. Since the hyperplane supporting the facet $\pi(F)$ is given by the equality $(f^V, x) = 0$ for $x \in \pi(\mathbb{R}^{E \cup E_0})$, we have $f' = 0$. Besides, obviously, $\pi(F) \cap \mathbb{R}^E = F^V$. Hence, $\pi(F)$ is zero-lifting of F^V. $\square$

8 Cones $\psi(Con_{n+1})$

Note that basic vectors of the space $\mathbb{R}^{E \cup E_0}$ are $e_{(ij)}$ for $(ij) \in E$ and $e_{(0i)}$ for $(0i) \in E_0$. Since $\pi(e_0) = \sum_{i \in V} \pi(e_{(0i)}) = 0$, we have $\dim\pi(\mathbb{R}^{E_0}) = n - 1 = \dim Q_n^w$. Note that $\pi(\mathbb{R}^E) = \mathbb{R}^E$. Hence, there is a one-to-one bijection χ between the spaces $\pi(\mathbb{R}^{E \cup E_0})$ and Q_n.

We define this bijection $\chi : \pi(\mathbb{R}^{E \cup E_0}) \to Q_n$ as follows

$$\chi(\mathbb{R}^E) = \varphi(\mathbb{R}^E) = \mathbb{R}_s^{E^{\mathcal{O}}}, \text{ and } \chi(\pi(\mathbb{R}^{E_0})) = Q_n^w,$$

where

$$\chi(e_{(ij)}) = \varphi(e_{(ij)}) = e_{ij} + e_{ji}, \text{ and } \chi(\pi(e_{(0i)})) = \chi(e_{(0i)} - \frac{1}{n} e_0) = q(i),$$

where $q(i)$ is defined in (1).

Note that $(e_{ij}+e_{ji})^2 = 2 = 2e^2_{(ij)}$ and

$$(q(i),q(j)) = -2 = 2n((e_{(0i)}-\frac{1}{n}e_0),(e_{(0j)}-\frac{1}{n}e_0)),\ q^2(i) = 2(n-1) = 2n(e_{(0i)}-\frac{1}{n}e_0)^2.$$

Roughly speaking, the map χ is a homothety that extends vectors $e_{(0i)}-\frac{1}{n}e_0$ up to vectors $q(i)$ by the multiple $\sqrt{2n}$.

Setting $\psi = \chi\circ\pi$, we obtain a map $\psi : \mathbb{R}^{E\cup E_0} \to Q_n$ such that

$$\psi(e_{(ij)}) = e_{ij}+e_{ji} \text{ for } (ij)\in E,\ \psi(e_{(0i)}) = q(i) \text{ for } i\in V. \tag{10}$$

Now we show how a point $x = x^V + x^0 \in \mathbb{R}^{E\cup E_0}$ is transformed into a point $q = \psi(x) = \chi(\pi(x)) \in Q_n$. We have $\pi(x) = x^V + \pi(x^0)$, where, according to (8), $x^V = \sum_{(ij)\in E} x_{(ij)}e_{(ij)} \in \pi(\mathbb{R}^E) = \mathbb{R}^E$ and $\pi(x^0) = \sum_{i\in V} x_{(0i)}(e_{(0i)}-\frac{1}{n}e_0) \in \pi(\mathbb{R}^{E_0})$. Obviously, $\chi(x^V+\pi(x^0)) = \chi(x^V)+\chi(\pi(x^0))$, and

$$\psi(x^V) = \chi(x^V) = \sum_{(ij)\in E} x_{(ij)}(e_{ij}+e_{ji}) = \varphi(x^V) = q^s \text{ and } \chi(\pi(x^0)) = \sum_{i\in V} x_{(0i)}q(i) = q^a.$$

Recall that $q^s = \sum_{(ij)\in E} q_{(ij)}(e_{ij}+e_{ji})$ and $q^a = \sum_{i\in V} w_i q(i)$. Hence

$$q_{(ij)} = x_{(ij)},\ (ij)\in E, \text{ and } w_i = x_{(0i)}. \tag{11}$$

Let $f \in \mathbb{R}^{E\cup E_0}$ be a facet vector of a facet F of the cone Con_{n+1}, $f = f^V + f^0 = \sum_{(ij)\in E} f_{(ij)}e_{(ij)} + \sum_{i\in V} f_{(0i)}e_{(0i)}$.

Let $(f,x)\ge 0$ be the inequality determining the facet F. The inequality $(f,x)\ge 0$ takes on the set $V\cup\{0\}$ the following form

$$(f,x) = \sum_{(ij)\in E} f_{(ij)}x_{(ij)} + \sum_{i\in V} f_{(0i)}x_{(0i)} \ge 0.$$

Since $x_{(ij)} = q_{(ij)}$, $x_{(0i)} = w_i$, we can rewrite this inequality as follows

$$(f,q) = (f^V,q^s)+(f^0,q^a) \equiv \sum_{(ij)\in E} f_{(ij)}q_{(ij)} + \sum_{i\in V} f_{(0i)}w_i \ge 0. \tag{12}$$

Comparing the inequality (12) with (7), we see that a canonical form of the facet vector f is $f = f^s + f^a$, where

$$f^s_{(ij)} = f_{(ij)},\ \text{for } (ij)\in E,\ f^a_{ij} = v_i - v_j \text{ where } v_i = \frac{1}{n}f_{(0i)},\ i\in V. \tag{13}$$

Theorem 2 *Let F be a facet of the cone Con_{n+1}. Then $\psi(F)$ is a facet of the cone $\psi(Con_{n+1})$ if and only if the facet F contains the extreme ray l_0 spanned by the vector e_0.*

Let $l\neq l_0$ be be an extreme ray of Con_{n+1}. Then $\psi(l)$ is an extreme ray of $\psi(Con_{n+1})$ if and only if the ray l belongs to a facet containing the extreme ray l_0.

Proof. By Theorem 1, the projection π transforms the facet F of Con_{n+1} into a facet of $\pi(Con_{n+1})$ if and only if $l_0 \subset F$. By the same Theorem, the projection $\pi(l)$ is an extreme ray of $\pi(Con_{n+1})$ if and only if l belongs to a facet containing the extreme ray l_0.

Recall that the map χ is a bijection between the spaces $\mathbb{R}^{E\cup E_0}$ and Q_n. This implies the assertion of this Theorem for the map $\psi = \chi \circ \pi$. $\square$

By Theorem 2, the map ψ transforms the facet F in a facet of the cone $\psi(Con_{n+1})$ only if F contains the extreme ray l_0, i.e., only if the equality $(f, e_0) = 0$ holds. Hence, the facet vector f should satisfy the equality $\sum_{i\in V} f_{(0i)} = 0$.

The inequalities (12) give all facet-defining inequalities of the cone $\psi(Con_{n+1})$ from known facet-defining inequalities of the cone Con_{n+1}.

A proof of Proposition 2 below will be given later for each of the cones Met_{n+1}, Hyp_{n+1} and Cut_{n+1} separately.

Proposition 2 *Let F be a facet of Con_{n+1} with facet vector $f = f^V + f^0$ such that $(f^0, e_0) = 0$. Then Con_{n+1} has also a facet F^* with facet vector $f^* = f^V - f^0$.*

Proposition 2 implies the following important fact.

Proposition 3 *For $q = q^s + q^a \in \psi(Con_{n+1})$, the map $q = q^s + q^a \to q^* = q^s - q^a$ preserves the cone $\psi(Con_{n+1})$, i.e.*

$$(\psi(Con_{n+1}))^* = \psi(Con_{n+1}).$$

Proof. Let F be a facet of Con_{n+1} with facet vector f. By Proposition 2, if $\psi(F)$ is a facet of $\psi(Con_{n+1})$, then F^* is a facet of Con_{n+1} with facet vector f^*. Let $q \in \psi(Con_{n+1})$. Then q satisfies as the inequality $(f, q) = (f^V, q^s) + (f^0, q^a) \geq 0$ (see (12)) so the inequality $(f^*, q) = (f^V, q^s) - (f^0, q^a) \geq 0$. But it is easy to see that $(f, q) = (f^*, q^*)$ and $(f^*, q) = (f, q^*)$. This implies that $q^* \in \psi(Con_{n+1})$. $\square$

The assertion of the following Proposition 4 is implied by the equality $(\psi(Con_{n+1}))^* = \psi(Con_{n+1})$. Call a facet G of the cone $\psi(Con_{n+1})$ *symmetric*, if $q \in F$ implies $q^* \in F$, and call this facet *asymmetric*, if it is not symmetric.

Proposition 4 *Let $g \in Q_n$ be a facet vector of an asymmetric facet G of the cone $\psi(Con_{n+1})$, and let $G^* = \{q^* : q \in G\}$. Then G^* is a facet of $\psi(Con_{n+1})$, and g^* is its facet vector.*

Recall that Con_{n+1} has facets, that are zero-lifting of facets of Con_n. Call a facet G of the cone $\psi(Con_{n+1})$ *zero-lifting* of a facet F^V of Con_n if $G = \psi(F)$, where F is a facet of Con_{n+1} which is zero-lifting of F^V.

Proposition 5 *Let $g \in Q_n$ be a facet vector of a facet G of the cone $\psi(Con_{n+1})$. Then the following assertions are equivalent:*

(i) $g = g^*$;

(ii) the facet G is symmetric;

(iii) $G = \psi(F)$, where F is a facet of Con_{n+1} which is zero-lifting of a facet F^V of Con_n.

(iv) G is a zero-lifting of a facet F^V of Con_n.

Proof. (i)⇒(ii). If $g = g^*$, then $g = g^s$. Hence, $q \in G$ implies $(g,q) = (g^s,q) = (g^s,q^s) = (g,q^*) = 0$. This means that $q^* \in G$, i.e., G is symmetric.

(ii)⇒(i). By Proposition 3, the map $q \to q^*$ is an automorphism of $\psi(Con_{n+1})$. This map transforms a facet G with facet vector g into a facet G^* with facet vector g^*. If G is symmetric, then $G^* = G$, and therefore, $g^* = g$.

(iii)⇒(i). Let $f = f^V + f^0$ be a facet vector of a facet F of Con_{n+1} such that $f^0 = 0$. Then the facet F is zero-lifting of the facet $F^V = F \cap \mathbb{R}^E$ of the cone Con_n. In this case, f^V is also a facet vector of the facet $G = \psi(F)$ of $\psi(Con_{n+1})$. Obviously, $(f^V)^* = f^V$.

(iii)⇒(iv). This implication is implied by definition of zero-lifting of a facet of the cone $\psi(Con_{n+1})$.

(iv)⇒(i). The map χ induces a bijection between $\pi(F)$ and $\psi(F)$. Since $\pi(F)$ is zero-lifting of F^V, the facet vector of $\pi(F)$ belongs to $\mathbb{R}^E$. This implies that the facet vector g of $\psi(F)$ belongs to $\mathbb{R}^E$, i.e., $g^* = g$. □

The symmetry group of Con_{n+1} is the symmetric group Σ_{n+1} of permutations of indices (see [DL97]). The group Σ_n is a subgroup of the symmetry group of the cone $\psi(Con_{n+1})$. The full symmetry group of $\psi(Con_{n+1})$ is $\Sigma_n \times \Sigma_2$, where Σ_2 corresponds to the map $q \to q^*$ for $q \in \psi(Con_{n+1})$. By Proposition 4, the set of facets of $\psi(Con_{n+1})$ is partitioned into pairs G, G^*. But it turns out that there are pairs such that $G^* = \sigma(G)$, where $\sigma \in \Sigma_n$.

9 Projections of hypermetric facets

The metric cone Met_{n+1}, the hypermetric cone Hyp_{n+1} and the cut cone Cut_{n+1} lying in the space $\mathbb{R}^{E \cup E_0}$ have an important class of *hypermetric* facets, that contains the class of *triangular* facets.

Let b_i, $i \in V$, be integers such that $\sum_{i \in V} b_i = \mu$, where $\mu = 0$ or $\mu = 1$. Usually these integers are denoted as a sequence $(b_1, b_2, ..., b_n)$, where $b_i \geq b_{i+1}$. If, for some i, we have $b_i = b_{i+1} = ... = b_{i+m-1}$, then the sequence is shortened as $(b_1, ..., b_i^m, b_{i+m}, ..., b_n)$.

One relates to this sequence the following inequality of type $(b_1, ..., b_n)$

$$(f(b), x) = -\sum_{i,j \in V} b_i b_j x_{(ij)} \geq 0,$$

where $x = \{x_{(ij)}\} \in \mathbb{R}^E$ and the vector $f(b) \in \mathbb{R}^E$ has coordinates $f(b)_{(ij)} = -b_i b_j$. This inequality is called of *negative* or *hypermetric* type if in the sum $\sum_{i \in V} b_i = \mu$ we have $\mu = 0$ or $\mu = 1$, respectively.

The set of hypermetric inequalities on the set $V \cup \{0\}$ determines a hypermetric cone Hyp_{n+1}. There are infinitely many hypermetric inequalities for metrics on $V \cup \{0\}$. But it is proved in [DL97], that only finite number of these inequalities determines facets of Hyp_{n+1}. Since triangle inequalities are inequalities $(f(b), x) \geq 0$ of type $b = (1^2, 0^{n-3}, -1)$, the hypermetric cone Hyp_{n+1} is contained in Met_{n+1}, i.e., $Hyp_{n+1} \subset Met_{n+1}$ with equality for $n = 2$.

The hypermetric inequality $(f(b), x) \geq 0$ takes the following form on the set $V \cup \{0\}$.

$$-\sum_{i,j \in V \cup \{0\}} b_i b_j x_{(ij)} = -\sum_{(ij) \in E} b_i b_j x_{(ij)} - \sum_{i \in V} b_0 b_i x_{(0i)} \geq 0. \tag{14}$$

If we decompose the vector $f(b)$ as $f(b)=f^V(b)+f^0(b)$, then $f^V(b)_{(ij)}=-b_ib_j$, $(ij)\in E$, and $f^0(b)_{(0i)}=-b_0b_i$, $i\in V$.

Let, for $S\subset V$, the equality $\sum_{i\in S}b_i=0$ hold. Denote by b^S a sequence such that $b^S_i=-b_i$ if $i\in S$ and $b^S_i=b_i$ if $i\not\in S$. The sequence b^S is called *switching* of b by the set S.

The hypermetric cone Hyp_{n+1} has the following property (see [DL97]). If an inequality $(f(b),x)\ge 0$ defines a facet and $\sum_{i\in S}b_i=0$ for some $S\subset V\cup\{0\}$, then the inequality $(f(b^S),x)\ge 0$ defines a facet, too.

Proof of Proposition 2 for Hyp_{n+1}.

Consider the inequality (14), where $(f^0(b),e_0)=-\sum_{i\in V}b_0b_i=0$. Then $\sum_{i\in V}b_i=0$. Hence, the cone Hyp_{n+1} has similar inequality for b^V, where $b^V_i=-1$ for all $i\in V$. Hence, if one of these inequalities defines a facet, so does another. Obviously, $f^0(b^V)=-f^0(b)$. Hence, these facets satisfy the assertion of Proposition 2. $\square$

Theorem 3 *Let $(f(b),x)\ge 0$ define a hypermetric facet of a cone in the space $\mathbb{R}^{E\cup E_0}$. Then the map ψ transforms it either in a hypemetric facet if $b_0=0$ or in a distortion of a facet of negative type if $b_0=1$. Otherwise, the projection is not a facet.*

Proof. By Section 8, the map ψ transforms the hypermetric inequality (14) for $x\in\mathbb{R}^{E\cup E_0}$ into the following inequality

$$-\sum_{(ij)\in E}b_ib_jq_{(ij)}-b_0\sum_{i\in V}b_iw_i\ge 0$$

for $q=\sum_{(ij)\in E}q_{(ij)}\varphi(e_{(ij)})+\sum_{i\in V}w_iq(i)\in Q_n$.

Since $f(b)$ determines a hypermetric inequality, we have $b_0=1-\sum_{i\in V}b_i=1-\mu$. So, the above inequality takes the form

$$\sum_{(ij)\in E}b_ib_jq_{(ij)}\le(\mu-1)\sum_{i\in V}b_iw_i.$$

By Theorem 1, this facet is projected by the map ψ into a facet if and only if $(f(b),e_0)=0$, where $e_0=\sum_{i\in V}e_{(0i)}$. Hence, we have

$$(f(b),e_0)=\sum_{i\in V}f(b)_{(0i)}=-\sum_{i\in V}b_0b_i=-b_0\mu=(\mu-1)\mu.$$

This implies that the hypermetric facet-defining inequality $(f(b),x)\ge 0$ is transformed into a facet-defining inequality if and only if either $\mu=0$ and then $b_0=1$ or $\mu=1$ and then $b_0=0$. So, we have

if $\mu=1$ and $b_0=0$, then the above inequality is a usual hypermetric inequality in the space $\psi(\mathbb{R}^E)=\varphi(\mathbb{R}^E)=\mathbb{R}^{E^{\mathcal{O}}}_s$;

if $\mu=0$ and $b_0=1$, then the above inequality is the following distortion of an inequality of negative type

$$-\sum_{(ij)\in E}b_ib_jq_{(ij)}-\sum_{i\in V}b_iw_i\ge 0,\ \text{ where } \sum_{i\in V}b_i=0. \tag{15}$$

$\square$

Comparing (7) with the inequality (15), we see that a canonical facet vector $g(b)$ of a facet of $\psi(Hyp_{n+1})$ has the form $g(b) = g^s(b) + g^a(b)$, where $g_{ij}(b) = g_{(ij)}(b) + v_i - v_j$, and

$$g_{(ij)}(b) = -b_i b_j, \; v_i = -\frac{1}{n} b_i \text{ for all } i \in V.$$

Define a cone of weighted quasi-hypermetrics $WQHyp_n = \psi(Hyp_{n+1})$ We can apply Proposition 3, in order to obtain the following assertion.

Proposition 6 *The map $q \to q^*$ preserves the cone $WQHyp_n$, i.e.*

$$(WQHyp_n)^* = WQHyp_n.$$

In other words, if $q \in WQHyp_n$ has weights $w_i, i \in V$, then the cone $WQHyp_n$ has a point q^* with weights $-w_i, i \in V$. $\square$

10 Generalizations of metrics

The metric cone Met_{n+1} is defined in the space $\mathbb{R}^{E \cup E_0}$. It has an extreme ray which is spanned by the vector $e_0 = \sum_{i \in V} e_{(0i)} \in \mathbb{R}^{E_0}$. Facets of Met_{n+1} are defined by the following set of triangle inequalities, where $d \in Met_{n+1}$.

Triangle inequalities of the sub-cone Met_n that define facets of Met_{n+1} that are zero-lifting and contain e_0:

$$d_{(ik)} + d_{(kj)} - d_{(ij)} \geq 0, \text{ for } i, j, k \in V. \tag{16}$$

Triangle inequalities defining facets that are not zero-lifting and contain the extreme ray l_0 spanned by the vector e_0:

$$d_{(ij)} + d_{(j0)} - d_{(i0)} \geq 0 \text{ and } d_{(ij)} + d_{(i0)} - d_{(j0)} \geq 0, \text{ for } i, j \in V. \tag{17}$$

Triangle inequalities defining facets that do not contain the extreme ray l_0 and do not define facets of Met_n.

$$d_{(i0)} + d_{(j0)} - d_{(ij)} \geq 0, \text{ for } i, j \in V. \tag{18}$$

One can say that the cone $Met_n \in \mathbb{R}^{E_0}$ is lifted into the space $\mathbb{R}^{E \cup E_0}$ using restrictions (17) and (18). Note that the inequalities (17) and (18) imply the following inequalities of non-negativity

$$d_{(i0)} \geq 0, \text{ for } i \in V. \tag{19}$$

A cone defined by inequalities (16) and (19) is called by cone $WMet_n$ of *weighted* metrics (d, w), where $d \in Met_n$ and $w_i = d_{(0i)}$ for $i \in V$ are *weights*.

If weights $w_i = d_{(0i)}$ satisfy additionally to the inequalities (19) also the inequalities (17), then the weighted metrics (d, w) form a cone $dWMet_n$ of *down-weighted* metrics. If metrics have weights that satisfy the inequalities (19) and (18), then these metrics are called *up-weighted* metrics. See details in [DD10], [DDV11].

Above defined generalizations of metrics are functions on unordered pairs $(ij) \in E \cup E_0$. Generalizations of metrics as functions on ordered pairs $ij \in E^{\mathcal{O}}$ are called *quasi-metrics*.

The cone $QMet_n$ of quasi-metrics is defined in the space $\mathbb{R}^{E^{\mathcal{O}}}$ by non-negativity inequalities $q_{ij} \geq 0$ for all $ij \in E^{\mathcal{O}}$, and by triangle inequalities $q_{ij} + q_{jk} - q_{ik} \geq 0$ for all ordered triples ijk for each $q \in QMet_n$. Below we consider in $QMet_n$ a sub-cone $WQMet_n$ of weighted quasi-metrics.

11 Cone of weighted quasi-metrics

We call a quasi-metric q *weighted* if it belongs to the subspace $Q_n \subset \mathbb{R}^{E^{\mathcal{O}}}$. So, we define

$$WQMet_n = QMet_n \cap Q_n.$$

A *quasi-metric* q is called *weightable* if there are weights $w_i \geq 0$ for all $i \in V$ such that the following equalities hold

$$q_{ij} + w_i = q_{ji} + w_j$$

for all $i, j \in V$, $i \neq j$. Since $q_{ij} = q^s_{ij} + q^a_{ij}$, we have $q_{ij} + w_i = q^s_{ij} + q^a_{ij} + w_i = q^s_{ji} + q^a_{ji} + w_j$, i.e., $q^a_{ij} - q^a_{ji} = 2q^a_{ij} = w_j - w_i$, what means that, up to multiple $\frac{1}{2}$ and sign, the antisymmetric part of q_{ij} is $w_i - w_j$. So, weightable quasi-metrics are weighted.

Note that weights of a weighted quasi-metric are defined up to an additive constant. So, if we take weights non-positive, we obtain a weightable quasi-metric. Hence, sets of weightable and weighted quasi-metrics coincide.

By definition of the cone $WQMet_n$ and by symmetry of this cone, the triangle inequality $q_{ij} + q_{jk} - q_{ik} \geq 0$ and non-negativity inequality $q_{ij} \geq 0$ determine facets of the cone $WQMet_n$. Facet vectors of these facets are

$$t_{ijk} = e_{ij} + e_{jk} - e_{ik} \text{ and } e_{ij},$$

respectively. It is not difficult to verify that $t_{ijk}, e_{ij} \notin Q_n$. Hence, these facet vectors are not canonical. Below, we give canonical representatives of these facet vectors.

Let $T(ijk) \subset K^{\mathcal{O}}_n$ be a triangle of $K^{\mathcal{O}}_n$ with direct arcs ij, jk, ki and opposite arcs ji, kj, ik. Hence

$$f^{T(ijk)} = (e_{ij} + e_{jk} + e_{ki}) - (e_{ji} + e_{kj} + e_{ik}).$$

Proposition 7 *Canonical representatives of facet vectors t_{ijk} and e_{ij} are*

$$t_{ijk} + t^*_{ijk} = t_{ijk} + t_{kji}, \text{ and } g(ij) = (e_{ij} + e_{ji}) + \frac{1}{n}(q(i) - q(j)),$$

respectively.

Proof. We have $t_{ijk} - f^{T(ijk)} = e_{ji} + e_{kj} - e_{ki} = t_{kji} = t^*_{ijk}$. This implies that the facet vectors t_{ijk} and t_{kji} determine the same facet, and the vector $t_{ijk} + t_{kji} \in \mathbb{R}^{E^{\mathcal{O}}}_s$ is a canonical representative of facet vectors of this facet. We obtain the first assertion of Proposition.

Consider now the facet vector e_{ij}. It is more convenient to take the doubled vector $2e_{ij}$. We show that the vector

$$g(ij) = 2e_{ij} - \frac{1}{n} \sum_{k \in V - \{i,j\}} f^{T(ijk)}$$

is a canonical representative of the facet vector $2e_{ij}$. It is sufficient to show that $g(ij) \in Q_n$, i.e., $g_{kl}(ij) = g^s_{kl}(ij) + w_k - w_l$. In fact, we have $g_{ij}(ij) = 2 - \frac{n-2}{n} = 1 + \frac{2}{n}$, $g_{ji}(ij) = \frac{n-2}{n} = 1 - \frac{2}{n}$, $g_{ik}(ij) = -g_{ki}(ij) = \frac{1}{n}$, $g_{jk}(ij) = -g_{kj}(ij) = -\frac{1}{n}$, $g_{kk'}(ij) = 0$. Hence, we have

$$g^s(ij) = e_{ij} + e_{ji}, \ w_i = -w_j = \frac{1}{n}, \text{ and } w_k = 0 \text{ for all } k \in V - \{i,j\}.$$

These equalities imply the second assertion of Proposition. $\square$

Let τ_{ijk} be a facet vector of a facet of Met_n determined by the inequality $d_{(ij)} + d_{(jk)} - d_{(ik)} \geq 0$. Then $t_{ijk} + t_{kji} = \varphi(\tau_{ijk})$, where the map $\varphi : \mathbb{R}^E \to \mathbb{R}^{E^{\mathcal{O}}}_s$ is defined in Section 2. Obviously, a triangular facet is symmetric.

Recall that $q_{ij} = q_{(ij)} + w_i - w_j$ if $q \in WQMet_n$. Let $i, j, k \in V$. It is not difficult to verify that the following equalities hold:

$$q^s_{ij} + q^s_{jk} - q^s_{ik} = q_{ij} + q_{jk} - q_{ij} \geq 0. \tag{20}$$

Since $q^s_{ij} = q^s_{ji} = q_{(ij)}$, these inequalities show that the symmetric part q^s of the vector $q \in WQMet_n$ is a semi-metric. Hence, if $w_i = w$ for all $i \in V$, then the quasi-semi-metric $q = q^s$ itself is a semi-metric. This implies that the cone $WQMet_n$ contains the cone of semi-metric Met_n. Moreover, $Met_n = WQMet_n \cap \mathbb{R}^{E^{\mathcal{O}}}_s$.

Now we show explicitly how the map ψ transforms the cones Met_{n+1} and $dWMet_n$ into the cone $WQMet_n$; see also Lemma 1 (ii) in [DDV11].

Theorem 4 *The following equalities hold*

$$\psi(Met_{n+1}) = \psi(dWMet_n) = WQMet_n \text{ and } WQMet^*_n = WQMet_n.$$

Proof. All facets of the metric cone Met_{n+1} of metrics on the set $V \cup \{0\}$ are given by triangular inequalities $d_{(ij)} + d_{(ik)} - d_{(kj)} \geq 0$. They are hypermetric inequalities $(g(b), d) \geq 0$, where b has only three non-zero values $b_j = b_k = 1$ and $b_i = -1$ for some triple $\{ijk\} \subset V \cup \{0\}$. By Theorem 3, the map ψ transforms this facet into a hypermetric facet, i.e., into a triangular facets of the cone $\psi(Met_{n+1})$ if and only if $b_0 = 0$, i.e., if $0 \notin \{ijk\}$. If $0 \in \{ijk\}$, then, by the same theorem, the equality $b_0 = 1$ should be satisfied. This implies $0 \in \{jk\}$. In this case, the facet-defining inequality has the form (15), that in the case $k = 0$, is

$$q_{(ij)} + w_i - w_j \geq 0.$$

This inequality is the non-negativity inequality $q_{ij} \geq 0$.

If $b_i = 1, b_j = -1$ and $k = 0$, the inequality $d_{(ij)} + d_{j0)} - d_{(0i)} \geq 0$ is transformed into inequality

$$q_{(ij)} + w_j - w_i \geq 0, \text{ i.e., } q^*_{ij} \geq 0.$$

This inequality and inequalities (20) imply the last equality of this Theorem.

The inequalities (18) define facets F of Met_{n+1} and $dWMet_n$ that do not contain the extreme ray l_0. Hence, by Theorem 3, $\psi(F)$ are not facets of $WQMet_n$. But, recall that the cone $dWMet_n$ contains all facets of Met_{n+1} excluding facets defined by the inequalities (18). Instead of these facets, the cone $dWMet_n$ has facets G_i defined by the non-negativity equalities (19) with facet vectors $e_{(0i)}$ for all $i \in V$. Obviously all these facets do not contain the extreme ray l_0. Hence, by Theorem 2, $\psi(G_i)$ is not a facet of $\psi(dWMet_n)$. Hence, we have also the equality $WQMet_n = \psi(dWMet_n)$. $\square$

Remark. Facet vectors of facets of Met_{n+1} that contain the extreme ray l_0 spanned by the vector e_0 are $\tau_{ijk} = \tau^V_{ijk}$, $\tau_{ij0} = \tau^V + \tau^0$ and $\tau_{ji0} = \tau^V - \tau^0$, where $\tau^V = e_{(ij)}$ and $\tau^0 = e_{(j0)} - e_{(i0)}$. Hence, Proposition 2 is true for Met_{n+1}, and we can apply Proposition 3 in order to obtain the equality $WQMet^*_n = WQMet_n$ of Theorem 4.

12 The cone Cut_{n+1}

The cut vectors $\delta(S) \in \mathbb{R}^{E \cup E_0}$ for all $S \subset V \cup \{0\}$ span all extreme rays of the cut cone $Cut_{n+1} \subset \mathbb{R}^{E \cup E_0}$. In other words, Cut_{n+1} is the conic hull of all cut vectors. Since the cone Cut_{n+1} is full-dimensional, its dimension is dimension of the space $\mathbb{R}^{E \cup E_0}$, that is $\frac{n(n+1)}{2}$.

Recall that $\delta(S) = \delta(V \cup \{0\} - S)$. Hence, we can consider only S such that $S \subset V$, i.e., $0 \notin S$. Moreover, by Section 5,

$$\delta(S) = \sum_{i \in S, j \notin S} e_{(ij)} = \sum_{i \in S, j \in V - S} e_{(ij)} + \sum_{i \in S} e_{(0i)} = \delta^V(S) + \sum_{i \in S} e_{(0i)}, \tag{21}$$

where $\delta^V(S)$ is restriction of $\delta(S)$ on the space $\mathbb{R}^E = \psi(\mathbb{R}^E)$. Note that

$$\delta(V) = \delta(\{0\}) = \sum_{i \in V} e_{(0i)} = e_0.$$

Consider a facet F of Cut_{n+1}. Let f be facet vector of F. Set

$$R(F) = \{S \subset V : (f, \delta(S)) = 0\}.$$

For $S \in R(F)$, the vector $\delta(S)$ is called *root* of the facet F. By (21), for $S \in R(F)$, we have

$$(f, \delta(S)) = (f, \delta^V(S)) + \sum_{i \in S} f_{(0i)} = 0. \tag{22}$$

We represent each facet vector of Cut_{n+1} as $f = f^V + f^0$, where $f^V \in \mathbb{R}^E$ and $f^0 \in \mathbb{R}^{E_0}$.

The set of facets of the cone Cut_{n+1} is partitioned onto equivalence classes by *switchings* (see [DL97]). For each $S, T \subset V \cup \{0\}$, the switching by the set T transforms the cut vector

$\delta(S)$ into the vector $\delta(S\Delta T)$, where Δ is symmetric difference, i.e., $S\Delta T = S\cup T - S\cap T$. It is proved in [DL97] that if $T \in R(F)$, then $\{\delta(S\Delta T) : S \in R(F)\}$ is the set of roots of the switched facet $F^{\delta(T)}$ of Cut_{n+1}. Hence, $R(F^{\delta(T)}) = \{S\Delta T : S \in R(F)\}$.

Let F be a facet of Cut_{n+1}. Then F contains the vector $e_0 = \delta(V)$ if and only if $V \in R(F)$. Hence, Lemma 1 below is an extended reformulation of Proposition 2.

Lemma 1 *Let F be a facet of Cut_{n+1} such that $V \in R(F)$. Let $f = f^V + f^0$ be facet vector of F. Then the vector $f^* = f^V - f^0$ is facet vector of switching $F^{\delta(V)}$ of the facet F, and $V \in R(F^{\delta(V)})$.*

Proof. Since $V \in R(F)$, $F^{\delta(V)}$ is a facet of Cut_{n+1}. Since $S\Delta V = V - S = \overline{S}$, for $S \subset V$, we have

$$R(F^{\delta(V)}) = \{\overline{S} : S \in R(F)\}.$$

Since $\emptyset \in R(F)$, the set $\emptyset\Delta V = V \in R(F^{\delta(V)})$. Now, using (22), for $S \in R(F^{\delta(V)})$, we have

$$(f^*, \delta(S)) = ((f^V - f^0), \delta(S)) = (f^V, \delta^V(S)) - \sum_{i\in S} f_{(0i)}.$$

Note that $\delta^V(\overline{S}) = \delta^V(S)$, and, since $V \in R(F)$, $\delta(V) = \delta(\{0\})$, we have $(f, \delta(V)) = \sum_{i\in V} f_{(0i)} = 0$. Hence, $\sum_{i\in\overline{S}} f_{(0i)} = -\sum_{i\in S} f_{(0i)}$. It is easy to see, that $(f^*, \delta(S)) = (f, \delta(\overline{S}))$. Since $S \in R(F^{\delta(V)})$ if and only if $\overline{S} \in R(F)$, we see that f^* is a facet vector of $F^{\delta(V)}$. $\square$

The set of facets of Cut_{n+1} is partitioned into orbits under action of the permutation group Σ_{n+1}. But some permutation non-equivalent facets are equivalent under switchings. We say that two facets F, F' of Cut_{n+1} belong to the same *type* if there are $\sigma \in \Sigma_{n+1}$ and $T \subset V$ such that $\sigma(F') = F^{\delta(T)}$.

13 Cone $OCut_n$

Denote by $OCut_n \subset \mathbb{R}^{E^{\mathcal{O}}}$ the cone whose extreme rays are spanned by ocut vectors $c(S)$ for all $S \subset V$, $S \neq \emptyset, V$. In other words, let

$$OCut_n = \{c \in Q_n : c = \sum_{S\subset V} \alpha_S c(S),\ \alpha_S \geq 0\}.$$

Coordinates c_{ij} of a vector $c \in OCut_n$ are given in (6), where $w_i \geq 0$ for all $i \in V$. Hence, $OCut_n \subset Q_n$. Recall that

$$c(S) = \frac{1}{2}(\delta^{\mathcal{O}}(S) + \sum_{i\in S} q(i)), \tag{23}$$

where $\delta^{\mathcal{O}}(S) = \varphi(\delta^V(S))$. Note that $\delta^{\mathcal{O}}(\overline{S}) = \delta^{\mathcal{O}}(S)$ and $q(\overline{S}) = -q(S)$, where $\overline{S} = V - S$.

Denote by $Cut_n^{\mathcal{O}} = \varphi(Cut_n)$ the cone generated by $\delta^{\mathcal{O}}(S)$ for all $S \subset V$. The vectors $\delta^{\mathcal{O}}(S)$ for all $S \subset V$, $S \neq \emptyset, V$, are all extreme rays of the cone $Cut_n^{\mathcal{O}}$ that we identify with Cut_n embedded into the space $\mathbb{R}^{E^{\mathcal{O}}}$.

Lemma 2 *For $S \subset V$, the following equality holds:*

$$\psi(\delta(S)) = 2c(S).$$

Proof. According to Section 8, $\psi(\delta^V(S)) = \varphi(\delta^V(S)) = \delta^{\mathcal{O}}(S)$. Besides, $\psi(e_{(0i)}) = q(i)$ for all $i \in V$. Hence, using (21), we obtain

$$\psi(\delta(S)) = \psi(\delta^V(S)) + \sum_{i \in S} \psi(e_{(0i)}) = \varphi(\delta^V(S)) + \sum_{i \in S} q(i) = \delta^{\mathcal{O}}(S) + q(S).$$

Recall that $\psi(\delta(V)) = \psi(e_0) = \mathbf{0}$ and $c(V) = 0$. Hence, according to (23), we obtain

$$\psi(\delta(S)) = 2c(S), \text{ for all } S \subset V.$$

Lemma is proved. □

Theorem 5 *The following equalities hold:*

$$\psi(Cut_{n+1}) = OCut_n \text{ and } OCut_n^* = OCut_n.$$

Proof. Recall that the conic hull of vectors $\delta(S)$ for all $S \subset V$ is Cut_{n+1}. The conic hull of vectors $c(S)$ for all $S \subset V$ is the cone $OCut_n$. Since $\psi(\delta(V)) = c(V) = \mathbf{0}$, the first result follows.

The equality $OCut_n^* = OCut_n$ is implied by the equalities $c^*(S) = c(\overline{S})$ for all $S \subset V$.

By Lemma 1, the equality $OCut_n^* = OCut_n$ is the special case $Con_{n+1} = Cut_{n+1}$ of Proposition 3. □

14 Facets of $OCut_n$

Lemma 3 *Let F be a facet of Cut_{n+1}. Then $\psi(F)$ is a facet of $OCut_n$ if and only if $V \in R(F)$.*

Proof. By Theorem 2, $\psi(F)$ is a facet of $OCut_n$ if and only if $e_0 = \delta(V) \subset F$, i.e., if and only if $V \in R(F)$. □

For a facet G of $OCut_n$ with facet vector g, we set

$$R(G) = \{S \subset V : (g, c(S)) = 0\}$$

and call the vector $c(S)$ for $S \in R(G)$ by *root* of the facet G.

Note that $\delta(\emptyset) = \mathbf{0}$ and $c(\emptyset) = c(V) = \mathbf{0}$. Hence, $\emptyset \in R(F)$ and $\emptyset \in R(G)$ for all facet F of Cut_{n+1} and all facets G of $OCut_n$. The roots $\delta(\emptyset) = \mathbf{0}$ and $c(\emptyset) = c(V) = \mathbf{0}$ are called *trivial* roots.

Proposition 8 *For a facet F of Cut_{n+1}, let $G = \psi(F)$ be a facet of $OCut_n$. Then the following equality holds:*

$$R(G) = R(F).$$

Remark. We give two proofs of this equality. Both are useful.

First proof. According to Section 8, the map ψ transforms an inequality $(f, x) \geq 0$ defining a facet of Cut_{n+1} into the inequality (12) defining the facet $G = \psi(F)$ of $OCut_n$. Recall the the inequality (12) relates to the representation of vectors $q \in Q_n$ in the basis $\{\varphi(e_{ij}), q(i)\}$, i.e., $q = \sum_{(ij)\in E} q_{(ij)}\varphi(e_{(ij)}) + \sum_{i\in V} w_i q(i)$. Let $q = c(S)$ for $S \in R(G)$. Then, according to (23), we have $q_{(ij)} = \frac{1}{2}\delta^V_{(ij)}(S)$, $w_i = \frac{1}{2}$ for $i \in S$ and $w_i = 0$ for $i \in \overline{S}$. Hence, omitting the multiple $\frac{1}{2}$, the inequality in (12) gives the following equality

$$\sum_{(ij)\in E} f_{(ij)}\delta^V_{(ij)}(S) + \sum_{i\in S} f_{(0i)} = 0$$

which coincides with (22). This implies the assertion of this Proposition.

Second proof. By Theorem 2, $\psi(l)$ is an extreme ray of $\psi(F)$ if and only if l is an extreme ray of F and $l \neq l_0$. Since l is spanned by $\delta(S)$ for some $S \in R(F)$ and $\psi(l)$ is spanned by $\psi(\delta(S)) = c(S)$, we have $R(G) = \{S \subset V : S \in R(F)\}$. Since $c(V) = \mathbf{0}$, we can suppose that $V \in R(G)$, and then $R(G) = R(F)$. $\square$

Remark. Note that $\delta(V) = \delta(\{0\}) = e_0 \neq \mathbf{0}$ is a non-trivial root of F, i.e., $V \in R(F)$. But $c(V) = \psi(\delta(V)) = \mathbf{0}$ is a trivial root of $R(G)$.

Recall that, for a subset $T \subset V$, we set $\overline{T} = V - T$. Note that $\overline{T} = V\Delta T$ and $\overline{T} \neq V \cup \{0\} - T$.

Lemma 4 *Let F be a facet of Cut_{n+1}, and $T \in R(F)$. Then the image $\psi(F^{\delta(T)})$ of the switched facet $F^{\delta(T)}$ is a facet of $OCut_n$ if and only if $\overline{T} \in R(F)$.*

Proof. By Lemma 3, $\psi(F^{\delta(T)})$ is a facet of $OCut_n$ if and only if $V \in R(F^{\delta(T)})$, i.e., if and only if $V\Delta T = \overline{T} \in R(F)$. $\square$

For a facet G of $OCut_n$, define $G^{\delta(T)}$ as the conic hull of $c(S\Delta T)$ for all $S \in R(G)$. Since each facet G of $OCut_n$ is $\psi(F)$ for some facet F of Cut_{n+1}, Lemma 4 and Proposition 8 imply the following assertion.

Theorem 6 *Let G be a facet of $OCut_n$. Then $G^{\delta(T)}$ is a facet of $OCut_n$ if and only if $T, \overline{T} \in R(G)$, and then $R(G^{\delta(T)}) = \{S\Delta T : S \in R(G)\}$.* $\square$

Theorem 6 asserts that the set of facets of the cone $OCut_n$ is partitioned onto equivalence classes by switchings $G \to G^{\delta(T)}$, where $T, \overline{T} \in R(G)$.

The case $T = V$ in Theorem 6 plays a special role. Recall that $V \in R(F)$ if F is a facet of Cut_{n+1} such that $\psi(F)$ is a facet of $OCut_n$. Hence, Lemma 1 and Proposition 3 imply the following fact.

Proposition 9 *Let F be a facet of Cut_{n+1} such that $\psi(F)$ is a facet of $OCut_n$. Let $g = g^s + g^a$ be a facet vector of the facet $\psi(F)$. Then the vector $g^* = g^s - g^a$ is a facet vector of the facet $\psi(F^{\delta(V)}) = (\psi(F))^* = (\psi(F))^{\delta(V)}$ such that $R((\psi(F))^*) = \{\overline{S} : S \in R(F)\}$.* $\square$

Recall that roughly speaking $OCut_n$ is projection of Cut_{n+1} along the vector $\delta(V) = \delta(\{0\})$.

Let $\sigma \in \Sigma_n$ be a permutation of the set V. For a vector $q \in \mathbb{R}^{E^{\mathcal{O}}}$, we have $\sigma(q)_{ij} = q_{\sigma(i)\sigma(j)}$. Obviously if g is a facet vector of a facet G of $OCut_n$, then $\sigma(g)$ is the facet vector of the facet $\sigma(G) = \{\sigma(q) : q \in G\}$.

Note that, by Proposition 9, the switching by V is equivalent to the operation $q \to q^*$. Hence, the symmetry group of $OCut_n$ contains the group $\Sigma_n \times \Sigma_2$, where Σ_2 relates to the map $q \to q^*$ for $q \in OCut_n$.

Theorem 7 *The group $\Sigma_n \times \Sigma_2$ is the symmetry group of the cone $OCut_n$.*

Proof. Let γ be a symmetry of $OCut_n$. Then γ is a symmetry of the set $\mathcal{F}(e_0)$ of facets F of the cone Cut_{n+1} containing the vector e_0. The symmetry group $\Gamma(e_0)$ of the set $\mathcal{F}(e_0)$ is a subgroup of the symmetry group of the cut-polytope $Cut^{\square}_{n+1}$. In fact, $\Gamma(e_0)$ is stabilizer of the edge e_0 of the polytope $Cut^{\square}_{n+1}$. But it is well-known that $\Gamma(e_0)$ consists of the switching by V and permutations $\sigma \in \Sigma_{n+1}$ leaving the edge e_0 non-changed. The map ψ transforms these symmetries of $\mathcal{F}(e_0)$ into symmetries $\sigma \in \Sigma_n$ and $q \to q^*$ of the cone $OCut_n$. $\square$

The set of all facets of $OCut_n$ is partitioned onto orbits of facets that are equivalent by the symmetry group $\Sigma_n \times \Sigma_2$. It turns out that, for some facets G, subsets $S \in R(G)$ and permutations $\sigma \in \Sigma_n$, we have $G^{\delta(S)} = \sigma(G)$.

By Proposition 5, if a facet of Cut_{n+1} is zero-lifting of a facet F^V of Cut_n, then the facet $G = \psi(F)$ of $OCut_n$ is symmetric and $G = G^* = G^{\delta(V)}$ is zero-lifting of F^V.

So, there are two important classes of orbits of facets of $OCut_n$. Namely, the orbits of symmetric facets, that are zero-lifting of facets of Cut_n, and orbits of asymmetric facets that are ψ-images of facets of Cut_{n+1} and are not zero-lifting.

15 Cases $3 \le n \le 6$

Compare results of this Section with Table 2 of [DDV11].

Most of described below facets are hypermetric or negative type. We give here the corresponding vectors b in accordance with Section 9.

n=3. Note that $Cut_4 = Hyp_4 = Met_4$. Hence,

$$OCut_3 = WQHyp_3 = WQMet_3.$$

All these cones have two orbits of facets: one orbit of non-negativity facets with $b = (1, 0, -1)$ and another orbit of triangular facets with $b = (1^2, -1)$.

n=4. We have $Cut_5 = Hyp_5 \subset Met_5$. Hence,

$$OCut_4 = WQHyp_4 \subset WQMet_4.$$

The cones $Hyp_5 = Cut_5$ have two orbits of facets: triangular and pentagonal facets. Recall that a triangular facet with facet vector τ_{ijk} is zero-lifting if $0 \notin \{ijk\}$. Hence, the cones $WQHyp_4 = OCut_4$ have three orbits of facets: of non-negativity with $b = (1, 0^2, -1)$, triangular with $b = (1^2, 0, -1)$ and weighted version of negative type with $b = (1^2, -1^2)$.

n=5. We have again $Cut_6 = Hyp_6 \subset Met_6$. Hence,

$$OCut_5 = WQHyp_5 \subset WQMet_5.$$

The cones $Hyp_6 = Cut_6$ have four orbits of facets, all are hypermetric: triangular with $b = (1^2, 0^3, -1)$, pentagonal with $b = (1^3, 0, -1^2)$ and two more types, one with $b = (2, 1^2, -1^3)$ and its switching with $b = (1^4, -1, -2)$. These four types provide 6 orbits of facets of the cones $WQHyp_5 = OCut_5$: non-negativity with $b = (1, 0^3, -1)$, triangular with $b = (1^2, 0^2, -1)$, of negative type with $b = (1^2, 0, -1^2)$, pentagonal with $b = (1^3, -1^2)$, and two of negative type with $b = (2, 1, -1^3)$ and $b = (1^3, -1, -2)$.

The last two types belong to the same orbit of the full symmetry group $\Sigma_5 \times \Sigma_2$. Hence, the cone $OCut_5$ has 5 orbits of facets under action of its symmetry group.

n=6. Now, we have $Cut_7 \subset Hyp_7 \subset Met_7$. Hence

$$OCut_6 \subset WQHyp_6 \subset WQMet_6.$$

The cone Cut_7 has 36 orbits of facets that are equivalent under action of the permutation group Σ_7. Switchings contract these orbits into 11 types F_k, $1 \le k \le 11$, of facets that are switching equivalent (see [DL97], Sect. 30.6). J.Vidali compute orbits of facets of $OCut_6$ under action of the group Σ_6. Using these computations, we give in Table below numbers of orbits of facets of cones Cut_7 and $OCut_6$ (cf. Figure 30.6.1 of [DL97]).

The first row of Table gives types of facets of Cut_7. In the second row of Table, for each type F_k, numbers of orbits of facets of Cut_7 of type F_k under action of the group Σ_7. The third row of Table, for each type F_k, gives numbers of orbits of facets of $OCut_6$ that are obtained from facets of type F_k under action of the group Σ_6. The fourth row gives, for each type F_k, numbers of orbits of facets of $OCut_6$ that are obtained from facets of type F_k under action of the group $\Sigma_6 \times \Sigma_2$.

The last column of Table gives total numbers of orbits of facets of the cones Cut_7 and $OCut_6$.

Table.

types	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	$\|\Omega\|$
Σ_7	1	1	2	1	3	2	4	7	5	3	7	36
Σ_6	2	2	4	1	3	2	7	13	6	6	15	61
$\Sigma_6 \times \Sigma_2$	2	2	3	1	2	1	4	7	3	4	8	37

The first three types F_1, F_2, F_3 relate to 4 orbits of hypermetric facets $F(b)$ of Cut_7 that are zero-lifting, where $b = (1^2, 0^4, -1)$, $b = (1^3, 0^2, -1^2)$ and $b = (2, 1^2, 0, -1^3)$, $b = (1^4, 0, -1, -2)$. Each of these four orbits of facets of Cut_7 under action of Σ_7 gives two orbits of facets of $OCut_6$ under action of the group Σ_6.

The second three types F_4, F_5, F_6 relate to 6 orbits of hypermetric facets $F(b)$ of Cut_7 that are not zero-lifting. Each of these 6 orbits gives one orbit of facets of $OCut_6$ under action of the group Σ_6.

The third three types F_7, F_8, F_9 relate to 16 orbits of facets of clique-web types $CW_1^7(b)$. These 16 orbits give 26 orbits of facets of $OCut_6$ under action of Σ_6.

The last two types $F_{10} = Par_7$ and Gr_7 are special (see [DL97]). They relate to 10 orbits of Cut_7, that give 21 orbits of facets of $OCut_6$ under action of Σ_6.

The subgroup Σ_2 of the full symmetry group $\Sigma_6 \times \Sigma_2$ contracts some pairs of orbits of the group Σ_6 into one orbit of the full group. The result is given in the forth row of Table.

Note that the symmetry groups of Cut_7 and $OCut_6$ have 36 and 37 orbits of facets, respectively.

References

[AACMP97] O.Aichholzer, F.Auerenhammer, D.Z.Chen, D.T.Lee, A.Mukhopadhyay and E.Papadopoulou, *Voronoi diagrams for direction-sensitive distances*, Proceedings of 13^{th} Annual ACM Simposium Computational Geometry, Nice, France (1997), 418–420.

[Aig79] M.Aigner, *Combinatorial Theory*, Springer-Verlag, Berlin 1979, (Chapter VII 3B).

[AM11] D.Avis and C.Meagher, *On the Directed Cut Cone and Polytope*, Manuscript Draft No. MAPR-D-11-00057, 2011.

[Bir67] G.Birkhoff, *Lattice Theory*, AMS, Providence, Rhode Island, 1967.

[CMM06] M.Charikar, K.Makarychev, and Y.Makarychev, *Directed metrics and directed graph partitioning problem* Proc. of 17th ACM-SIAM Symposium on Discrete Algorithms (2006) 51–60.

[DD10] M.Deza and E.Deza, *Cones of Partial Metrics*, Contributions in Discrete Mathematics, **6** (2010), 26–41.

[DDV11] M.Deza, E.Deza and J.Vidali, *Cones of Weighted and Partial Metrics*, arXiv: 1101.0517v2[math.Co]04 Jan 2011, to appear in *Algebra 2010: Advances in Algebraic Structures*, World Scientific, 2011.

[DL97] M.Deza and M.Laurent, *Geometry of cuts and metrics*, Springer-Verlag, 1997.

[Ha14] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, Verlag "Veit and Co", 1914.

[Se97] A.K.Seda, *Quasi-metrics and semantic of logic programs*, Fundamenta Informaticae, **29** (1997), 97–117.